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A BAND ABSORPTANCE FORMULATION FOR NONISOTHERMAL GASEOUS RADIATION*

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Abstract—A method is presented by which correlations for the total band absorptance of vibration-rotation bands, applicable to isothermal gases, may be extended to predict the total band absorptance for nonisothermal gases. The method is analogous to the two-parameter Curtis–Godson approximation, and it consists of formulating the total band absorptance in terms of three parameters which describe an equivalent isothermal gas. These three parameters are chosen such that the nonisothermal band absorptance is correct in the linear, square-root and logarithmic limits. The formulation is shown to be in reasonable agreement with experimental data and other analytical predictions.

NOMENCLATURE

A,	total band absorptance [cm^{-1}];
A_0,	band width parameter, $\sqrt{(T/\alpha)}$ for rigid rotor and harmonic oscillator [cm^{-1}];
$\bar{A}$,	dimensionless band absorptance;
B_e,	rotational constant;
c,	speed of light;
d,	mean line spacing [cm^{-1}];
h,	Planck's constant;
k,	Boltzmann's constant;
L,	path length [cm];
P,	pressure [atm];
S,	total band intensity [$\text{atm}^{-1}\,\text{cm}^{-2}$];
S_j,	line intensity [$\text{atm}^{-1}\,\text{cm}^{-2}$];
T,	temperature [°K];
T_0,	maximum gas temperature;
T_1,	minimum gas temperature;
u,	dimensionless path length, SPL/A_0;
y,	path length coordinate [cm];
y^*,	dimensionless coordinate;
α,	$hc/4kB_e$;
β,	line structure parameter, $4\gamma/d$;
γ,	mean line half width [cm^{-1}];
θ,	T_0/T_1;
κ_ω,	spectral absorption coefficient [cm^{-1}];
ξ,	$\alpha\omega^2/T_0$;
ϕ,	T_0/T;
ω,	wave number [cm^{-1}].

INTRODUCTION

THE PURPOSE of the present paper is to formulate a method of predicting the total band absorptance for vibration–rotation bands when the radiating gas is nonisothermal. Specifically, a method is presented by which the total band absorptance for nonisothermal gases may be formulated in terms of the corresponding band absorptance for an isothermal gas; that is, the nonisothermal gas is recast as an equivalent isothermal gas. This approach is, in essence, a total band extension of the well-known Curtis–Godson method.

The Curtis–Godson approximation (see for example Goody [1], pp. 236–243) consists of introducing equivalent path length and line structure parameters, such that the band absorptance for a nonhomogeneous gas may be formulated in terms of an equivalent homogeneous gas. These two equivalent parameters are

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obtained by comparing the nonhomogeneous and homogeneous band absorptance expressions in the weak-line and strong-line limits, with the Curtis–Godson approximation constituting an interpolation between these two exact results. When the nonhomogeneity is due solely to variable pressure, the Curtis–Godson approximation may be applied to a total vibration–rotation band. However, for nonisothermal gases the Curtis–Godson approximation is directly applicable only to narrow spectral regions of a band, since the line intensity is assumed to be independent of wave number.

Several investigators have applied the Curtis–Godson approximation to a total band, with the most detailed treatment being that given by Weiner and Edwards [2]. This involved application of the Curtis–Godson approximation to each spectral interval of the band, and then numerically integrating over the entire band to obtain the total band absorptance. In this context, the results of [2] constitute an interpolation between the exact strong-line and overlapped-line limits.

Recently Chan and Tien [3] have also applied the Curtis–Godson approximation to each spectral interval of a band, but simplified the integration over the entire band by requiring that certain moments of the wave number integrals be satisfied rather than performing the wave number integrations directly. They also presented a second method which consisted of expanding the spectral absorption coefficient about the midpoint temperature of the gas.

The total band absorptance for an isothermal gas is characterized by three parameters (as opposed to two parameters for a narrow spectral region), and the present investigation is concerned with formulating the nonisothermal band absorptance in terms of three analogous parameters which describe an equivalent isothermal gas. These three equivalent parameters are chosen such that the nonisothermal band absorptance is correct in the linear, square-root and logarithmic limits. In effect, this constitutes a three-parameter total-band extension to the two-parameter narrow-band procedure of Curtis and Godson.

ANALYSIS

Taking the wave number origin at the center of a given vibration–rotation band, the total band absorptance for a nonisothermal gas is expressed as

$$A = \int_{-\infty}^{\infty} \{1 - \exp[-\int_{0}^{L} \kappa_{\omega}(y)\,dy]\}\,d\omega \qquad (1)$$

where L denotes the thickness of the gas layer. For present purposes it will be assumed that the individual rotational lines which comprise the vibration–rotation band are Lorentz lines, such that the spectral absorption coefficient for the jth line is given by

$$\frac{\kappa_{\omega j}}{P} = \frac{S_j}{\pi}\frac{\gamma}{\gamma^2 + (\omega - \omega_j)^2} \qquad (2)$$

where ω_j denotes the center of line j.

In actuality the line half width γ depends upon the rotational quantum number j, and for a single line $\gamma_j \sim 1/\sqrt{T}$. With reference to equation (2), however, it will be assumed that γ represents an appropriate mean half width, averaged over both rotational and vibrational energy levels. Thus the temperature dependence of γ will not necessarily be the same as that for a single line.

It will further be assumed that the distribution of line intensities is given by the rigid-rotor harmonic-oscillator model, for which the variation of line intensity with wave number is [4]

$$S_j = \frac{Sd\alpha\,|\omega|}{T}\exp\left(-\frac{\alpha\omega^2}{T}\right) \qquad (3)$$

where

$$\alpha = \frac{hc}{4kB_e}.$$

As will be discussed later, however, the final results are not necessarily dependent upon the use of equation (3).

With reference to an isothermal gas, the band absorptance may be expressed as a function of three parameters in the general form [5]

$$A = A_0\bar{A}(u, \beta) \qquad (4)$$

where $\bar{A}(u, \beta)$ is a dimensionless band absorptance, u is a dimensionless path length, $u = SPL/A_0$, and β denotes the line structure parameter defined as $\beta = 4\gamma/d$. The third parameter, A_0, is the band width parameter, and this is dependent upon the particular molecular model used. For the present case of a rigid rotor and harmonic oscillator, A_0 is expressed as $A_0 = \sqrt{(T/\alpha)}$.

Upon employing equations (2) and (3) in equation (1), Edwards and Menard [6] have shown that for an isothermal gas equation (4) possesses three limiting forms, and these are*:

1. Linear, $u \ll 1$;

$$A = A_0 u \qquad (5)$$

2. Square root, $\beta \ll 1$ and $u \gg \beta$;

$$A = 2^{\frac{3}{2}}\Gamma(3/4)\, A_0\sqrt{(u\beta)} \qquad (6)$$

3. Logarithmic, $u \gg 1$;

$$A = 2A_0\sqrt{(\ln u)}. \qquad (7)$$

Furthermore, in the Appendix it is shown that for a nonisothermal gas precisely the same limiting expressions are obtained for the band absorptance, providing A_0, u, and β are replaced by the equivalent parameters $\tilde{A}_0$, $\tilde{u}$, and $\tilde{\beta}$, which are defined by †

$$\tilde{A}_0 = A_0(T_0) \qquad (8)$$

$$\tilde{u} = \frac{P}{\tilde{A}_0}\int_0^L S(T)\,\mathrm{d}y \qquad (9)$$

$$\tilde{\beta} = \frac{P}{\tilde{u}\tilde{A}_0}\int_0^L \beta(T)S(T)\sqrt{(T/T_0)}\,\mathrm{d}y \qquad (10)$$

where T_0 denotes the maximum temperature within the gas.*

It is apparent that in the three limits the nonisothermal quantities $\tilde{A}_0$, $\tilde{u}$ and $\tilde{\beta}$ correspond to A_0, u and β for an equivalent isothermal gas. Correspondingly, in the same spirit as the Curtis–Godson approximation, it will be assumed that the isothermal band absorptance given by equation (4) may be extended to nonisothermal gases by writing equation (4) as

$$A = \tilde{A}_0\bar{A}(\tilde{u}, \tilde{\beta}). \qquad (11)$$

The motivation for this assumption is, of course, the fact that equation (11) is correct in each of the three limits.

It is worth mentioning that equation (11) together with equations (8)–(10) are not restricted to the molecular model of a rigid rotor and harmonic oscillator. For example, if the variation of line intensity with wave number is taken to be that given by the exponential wide-band model of Edwards [6]

$$S_j = \frac{Sd}{A_0}\exp(-2\,|\omega|/A_0)$$

instead of equation (3), one still obtains equations (8)–(10) for the equivalent nonisothermal parameters. In other words, the use of equations (8)–(10) in extending an isothermal band absorptance correlation to include nonisothermal gases is not necessarily restricted to the line intensity distribution of equation (3).

It is interesting to compare the present three-parameter approximation with the Curtis–Godson two-parameter approximation. The equivalent path length $\tilde{u}$, which is determined from the linear limit, is exactly that given by the Curtis–Godson approximation, since in the linear limit the spacial and wave number integrations may be interchanged, and correspondingly the restriction to a narrow band is not necessary. The equivalent line structure parameter $\tilde{\beta}$, as given by equation (10), differs,

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* The linear limit is completely general, whereas the expressions for the square-root and logarithmic limits, given by equation (6) and (7), apply only to the rigid-rotor harmonic-oscillator model.

† Equations (9) and (10) may readily be extended to include nonuniform pressure as well as nonuniform temperature.

* Recently D. K. Edwards and S. J. Morizumi of UCLA independently derived equation (10), but they were led to a different result for $\tilde{A}_0$.

however, from that of the Curtis–Godson approximation through the inclusion of $\sqrt{(T/T_0)}$ in the integral of equation (10). This difference in the weighting function is due to the use of wave-number dependent line intensities in the present analysis (total band), while in the Curtis–Godson approximation the line intensities are assumed to be independent of wave number (narrow band). In both cases, the equivalent line structure parameter is obtained from the square-root (strong-line) limit.

The third equivalent parameter, $\tilde{A}_0$, does not have a counterpart in the two parameter Curtis–Godson approximation, since a narrow band model utilizes a fixed wave number interval in place of A_0 in equation (4). Unlike the other two equivalent parameters, $\tilde{A}_0$ does not represent an averaged quantity. Instead, as shown by equation (8), $\tilde{A}_0$ is simply the band-width parameter for an isothermal gas evaluated at the maximum temperature within the non-isothermal gas. This is physically quite acceptable. In the Appendix it is shown that $\tilde{A}_0$ is obtained from the logarithmic limit. Furthermore, this limit corresponds to the physical situation in which the central portion of the band is saturated. Since the width of a vibration–rotation band increases with temperature, this means that the band spectrum for colder regions of the gas will be confined to wave numbers for which complete absorption occurs in the highest temperature regions. The band wings for the region of highest temperature will, in turn, correspond to wave numbers for which the colder gas is transparent. Correspondingly, in the logarithmic limit one would anticipate that the absorption of incident radiation, and thus the total band absorptance, should be governed by the region of highest temperature within the gas, and this is consistent with the present definition for the equivalent band width parameter $\tilde{A}_0$.

COMPARISON WITH NUMERICAL RESULTS

In order to compare equation (11) with an exact numerical calculation, consider the limit of overlapped lines ($\beta \gg 1$). In addition, attention will be given to a fundamental band, for which

$$S(T) = \frac{T_0}{T} S(T_0) \tag{12}$$

together with the family of temperature profiles

$$\frac{T}{T_1} = [1 + (\theta^n - 1)\, y^*]^{1/n} \tag{13}$$

where $y^* = y/L$ and $\theta = T_0/T_1$. Equation (13) is illustrated graphically in Fig. 1 for $\theta = 2$.

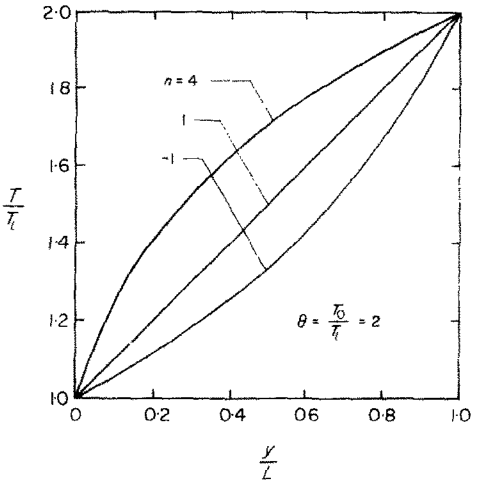

FIG. 1. Illustrative temperature profiles.

Noting that for overlapped lines [4]

$$\frac{\kappa_\omega}{P} = \frac{S_j}{d}$$

and upon combining equations (1), (3) and (12), the nonisothermal band absorptance may be expressed as

$$A = 2A_0(T_0) \int_0^\infty \{1 - \exp[-u_0 \xi G(\xi, \theta)]\}\, \mathrm{d}\xi \tag{14}$$

where

$$G(\xi, \theta) = \frac{n\theta^n}{(\theta^n - 1)} \int_1^\theta \phi^{1-n} \mathrm{e}^{-\xi^2 \phi}\, \mathrm{d}\phi \tag{15}$$

and

$$u_0 = \frac{S(T_0)}{A_0(T_0)} PL, \quad \xi^2 = \frac{\alpha\omega^2}{T_0}, \quad \phi = T_0/T.$$

With respect to the approximate equation (11), the corresponding isothermal band absorptance yields

$$A = 2\tilde{A}_0 \int_0^\infty [1 - \exp(-\tilde{u}\xi\, e^{-\xi^2})]\, d\xi \qquad (16)$$

where, from equations (9) and (13)

$$\tilde{u} = \frac{u_0 \theta n}{n-1}\left(\frac{\theta^{n-1} - 1}{\theta^n - 1}\right).$$

Equations (14) and (16) have been evaluated numerically for $n = 4$, 1 and -1, and a comparison between the exact result of equation (14) and the approximate formulation of equation (16) is illustrated in Fig. 2. The maximum discrepency occurs for intermediate values of u_0, which is as expected, since the approximate band absorptance becomes exact for either $u_0 \to 0$ or $u_0 \to \infty$.

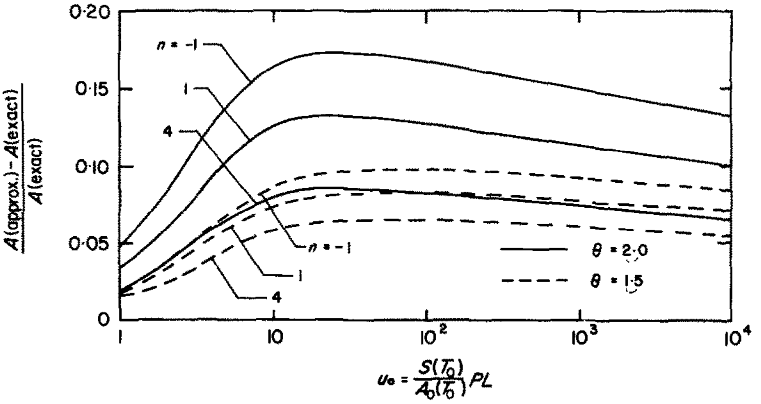

FIG. 2. Comparison of exact and approximate band absorptance results for overlapped lines.

As previously mentioned, when the Curtis–Godson approximation is applied to each spectral interval, with subsequent integration over the entire band, as in the work of Weiner and Edwards [2], the result is exact in the overlapped-line limit. Thus, Fig. 2 is indicative of the additional error involved in using equation (11) as opposed to the spectral application of the Curtis–Godson approximation. On the other hand, the present method enjoys the advantage of not requiring detailed spectral information nor of requiring lengthy numerical integrations.

COMPARISON WITH OTHER RESULTS

Both experimental and analytical nonisothermal band absorptance results are available for the 2·7 μ band of H_2O. With reference to the present analysis, an appropriate band absorptance correlation for isothermal gases is that of Tien and Lowder [5], such that from equation (11) one has for a nonisothermal gas

$$A = \tilde{A}_0 \ln\left\{\tilde{u} f(\bar{\beta})\left[\frac{\tilde{u} + 2}{\tilde{u} + 2f(\bar{\beta})}\right] + 1\right\} \qquad (17)$$

with

$$f(\bar{\beta}) = 2·94[1 - \exp(-2·60\bar{\beta})].$$

Furthermore, for the 2·7 μ H_2O band, expressions for $S(T)$, $A_0(T)$, and $\beta(T)$ are given by Edwards *et al.* [7] for a nonisothermal gas as

$$S(T) = 171\left(\frac{300}{T}\right), \text{atm}^{-1}\,\text{cm}^{-2} \tag{18a}$$

$$A_0(T) = 113\left(\frac{T}{300}\right)^{\frac{1}{2}}, \text{cm}^{-1} \tag{18b}$$

$$\beta(T) = 0{\cdot}145\,P_e\left(\frac{300}{T}\right)^{\frac{1}{2}} \tag{18c}$$

where P_e is the equivalent broadening pressure [7]. It readily follows from equations (9) and (10) that

$$\tilde{u} = \frac{PS(T_0)}{\tilde{A}_0}\int\limits_0^L \frac{T_0}{T(y)}\,\mathrm{d}y, \quad \tilde{\beta} = \beta(T_0)$$

and, with $\tilde{A}_0 = A_0(T_0)$, this completely describes equation (17) for application to a nonisothermal gas.

The present analysis is compared in Table 1 with the experimental results of Weiner and Edwards [2] as well as other analytical predictions. In the experiments, the temperature increased in a nearly linear manner from one end of the cell to a maximum temperature, and then decreased, again in a nearly linear manner, toward the other end of the cell.* The ratio of maximum to minimum temperature ranged from 2·3 to 2·9.

Table 1. *Band absorptance comparisons for the* 2·7 μ H_2O *band*

Run number	Band absorptance (cm^{-1}) 12–10	9–13	12–1
Weiner and Edwards [2] (experimental)	599	451	372
Weiner and Edwards [2] (analytical)	564	454	450
Chan and Tien [3]	545	456	440
Present analysis	613	498	468

As previously discussed, the analytical results of Weiner and Edwards [2] were obtained by applying the Curtis–Godson approximation to each spectral interval and then integrating numerically over the entire band. In addition, separate treatment was given to each of the three bands which comprise the composite 2·7 μ H_2O band. The results of Chan and Tien [3] utilized equations (17) and (18), and employed equivalent spectroscopic parameters based upon satisfying moments of wave number integrals over the entire composite band. The primary difference between the results of Chan and Tien and the present analysis thus lies in the method of evaluating the equivalent parameters. On the whole the present analysis, which has the advantage of relative simplicity, differs from the experimental results by a slightly greater amount than the other two analyses.

An additional comparison may be made with the experimental results of Edwards *et al* [7] for the two fundamental bands of CO_2. In this case the temperature varied monatonically from one end of the cell to the other, with the ratio of maximum to minimum temperature ranging from 3·4 to 3·7. For the CO_2 fundamental bands the line structure parameter varies with temperature as $\beta \sim \sqrt{T}$ [7] instead of inversely with the square root of temperature as for H_2O. It follows from equation (10) that

$$\tilde{\beta} = \beta(T_0)\left[\frac{1}{L}\int\limits_0^L \frac{T_0}{T(y)}\,\mathrm{d}y\right]^{-1}.$$

The isothermal band absorptance correlation and spectroscopic parameters of Edwards *et al.*

Table 2. *Comparison of results for* CO_2

Run number	Band	Band absorptance (cm^{-1}) Edwards *et al.* [7] (experimental)	Present analysis.
T10	15μ	253	242
T11	15μ	143	136
T15	15μ	43	48
T6	4·3μ	254	329
T5	4·3μ	189	241
T16	4·3μ	132	146

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* In evaluating $\tilde{u}$, the triangle-like temperature profiles employed in [3] were used.

[7] were employed in evaluating equation (11), and the comparison with experimental results is shown in Table 2. The average discrepency between experimental and analytical results is 14·7 per cent and it is doubtful that the isothermal band absorptance correlation is more accurate than this.

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APPENDIX

Limiting Solutions

Linear limit

Expanding the exponential in equation (1), and noting that

$$S = \int_{-\infty}^{\infty} \frac{\kappa_\omega}{P} \mathrm{d}\omega$$

there then follows that

$$A = \bar{A}_0 \tilde{u} \qquad \text{(A.1)}$$

where $\tilde{u}$ is defined by equation (9), and equation (A.1) is of the same form as equation (5). This is identical with the Curtis–Godson weak-line limit.

Logarithmic limit

Since line structure plays no role in the logarithmic limit, the overlapped line model may be employed to obtain this limit. Recalling that for overlapped lines [4]

$$\frac{\kappa_\omega}{P} = \frac{S_j}{d}$$

then upon combining equations (1) and (3), and letting

$$z = \frac{P}{\tilde{u}A_0(T_0)} \int_0^y S(T)\,\mathrm{d}y, \quad \phi(z) = \frac{T_0}{T(y)}$$

there is obtained

$$A = 2A_0(T_0) \int_0^\infty \{1 - \exp[-\tilde{u}\phi_m \xi \int_0^1 \exp(-\xi^2\phi(z))\,\mathrm{d}z]\}\,\mathrm{d}\xi \qquad \text{(A.2)}$$

where, by the mean-value theorem

$$\phi_m = \frac{\int_0^1 \phi(z) \exp[-\xi^2\phi(z)]\,\mathrm{d}z}{\int_0^1 \exp[-\xi^2\phi(z)]\,\mathrm{d}z}.$$

From equation (A.2) it may easily be shown that large $\tilde{u}$ corresponds to large ξ, and by the method of steepest descent [8] there exist the asymptotic expansions

$$\int_0^1 \exp[-\xi^2\phi(z)]\,\mathrm{d}z \sim -\frac{\mathrm{e}^{-\xi^2}}{\xi^2\phi'(z_0)}; \quad \phi'(z_0) \neq 0 \qquad \text{(A.3)}$$

$$\int_0^1 \exp[-\xi^2\phi(z)]\,\mathrm{d}z \sim \left[\frac{2\pi}{\phi''(z_0)}\right]^{\frac{1}{2}} \frac{\mathrm{e}^{-\xi^2}}{\xi}; \phi'(z_0) = 0 \qquad \text{(A.4)}$$

where z_0 denotes the position of minimum $\phi(z)$; i e. $\phi = 1$. It is clear that equation (A.3) applies when $\phi(z)$, and thus $T(z)$, is a monatonic function of z, whereas equation (A.4) is employed when $\phi(z)$ has a minimum (maximum temperature) at $z = z_0$. In either case, equation (A.2) may be written as

$$A = 2A_0(T_0) \int_0^\infty \{1 - \exp[-\tilde{u}\phi_m c \xi^{-n} \mathrm{e}^{-\xi^2}]\}\,\mathrm{d}\xi \qquad \text{(A.5)}$$

where c is evaluated from either equation (A.3) or equation (A.4), while $n = 1$ for the conditions of equation (A.3) and $n = 0$ with reference to equation (A.4). Following the procedure of Edwards and Menard [6], the limiting form of equation (A.5) for large $\tilde{u}$ is

$$A = 2A_0(T_0)\sqrt{(\ln\tilde{u})} = 2\bar{A}_0\sqrt{(\ln\tilde{u})} \qquad \text{(A.6)}$$

and it is thus unnecessary to evaluate either ϕ_m or c. Equation (A.6) coincides with the logarithmic limit for an isothermal gas given by equation (7).

It should be emphasized that the evaluation of A_0 at the maximum temperature, T_0, in equation (A.6) is not simply a consequence of employing T_0 as a reference temperature in defining the dimensionless quantities appearing in equation (A.2). If some arbitrary temperature had instead been used, equation (A.6) would still result.

Square-root limit

Let A_j denote the average absorptance of the jth line, such that

$$A_j = \frac{1}{d}\int_{-\infty}^{\infty}\left[1 - \exp\left(-\int_0^L \kappa_{\omega j}(y)\,\mathrm{d}y\right)\right]\mathrm{d}(\omega - \omega_j)\,. \tag{A.7}$$

Upon combining this with equation (2) and following the same procedure as outlined for an isothermal gas by Goody ([1] p. 126), the square-root limit for a nonisothermal gas is found to be

$$A_j = \frac{2}{d}\sqrt{\left(P\int_0^L S_j(y)(\beta y)\,\mathrm{d}y\right)}. \tag{A.8}$$

Since the square root limit implies nonoverlapping lines, then

$$A = \sum_j A_j d \simeq \int_{-\infty}^{\infty} A_j\,\mathrm{d}\omega \tag{A.9}$$

and upon employing equations (3) and (A.8)

$$A = 2\int_0^{\infty}\left[P\int_0^L \frac{1}{T}S(T)\,\beta(T)\,\alpha\omega \exp\left(-\alpha\omega^2/T\right)\mathrm{d}y\right]^{\frac{1}{2}}\mathrm{d}\omega. \tag{A.10}$$

Now, letting

$$\eta = \frac{P}{\bar{A}_0\tilde{u}\tilde{\beta}}\int_0^y S(T)\,\beta(T)\sqrt{\left(\frac{T}{T_0}\right)}\mathrm{d}y$$
$$\tilde{\beta} = \frac{P}{\bar{A}_0\tilde{u}}\int_0^L S(T)\,\beta(T)\sqrt{\left(\frac{T}{T_0}\right)}\mathrm{d}y \tag{A.11}$$

then from equation (A.10)

$$A = 2\bar{A}_0\sqrt{(\tilde{u}\tilde{\beta})}\int_0^{\infty}\left[\xi\int_0^1 \phi^{\frac{1}{2}}\,\mathrm{e}^{-\xi^2\phi}\,\mathrm{d}\eta\right]^{\frac{1}{2}}\mathrm{d}\xi. \tag{A.12}$$

Upon employing Schwarz's inequality, it is found that a lower limit to equation (A.12) exists which is independent of the temperature profile, and this is

$$A \geqslant 2^{\frac{3}{4}}\Gamma(3/4)\,\bar{A}_0\sqrt{(\tilde{u}\tilde{\beta})}. \tag{A.13}$$

The quantity on the right side of this equation is exactly the isothermal square-root limit as given by equation (6), with the isothermal parameters replaced by the equivalent nonisothermal quantities $\tilde{u}$, $\bar{A}_0$ and $\tilde{\beta}$

It appears, however, that equation (A.13) constitutes more than just a lower bound on the nonisothermal band absorptance. For example, if one assumes that the nonisothermal square-root limit can be described in terms of a reference temperature T_r, then upon substituting $\phi_r = T_0/T_r$ into equation (A.12), the reference temperature vanishes when the integrations are performed, and there is obtained

$$A = 2^{\frac{3}{4}}\Gamma(3/4)\,\bar{A}_0\sqrt{(\tilde{u}\tilde{\beta})} = 2{\cdot}06\bar{A}_0\sqrt{(\tilde{u}\tilde{\beta})} \tag{A.14}$$

which is the same as the lower bound given by equation (A.13).

In reality, the assumption of an appropriate reference temperature is not mathematically exact. Nevertheless, this suggests that equation (A.14) might well be a valid approximation to equation (A.12). To illustrate this, consider again the family of temperature profiles given by equation (13), and assume a hypothetical band for which* $S(T)\beta(T) \sim 1/T$. From equations (13), (A.10) and (A.11) the band absorptance may be expressed as

$$A = H(n, \theta)\,\bar{A}_0\sqrt{(\tilde{u}\tilde{\beta})} \tag{A.15}$$

where

$$H(n, \theta) = 2\left[\left(\frac{2n-1}{2n\theta^{\frac{1}{2}}}\right)\left(\frac{\theta^n - 1}{\theta^{n-\frac{1}{2}} - 1}\right)\right]^{\frac{1}{2}} \int_0^{\infty}\left[\xi G(\xi, \theta)\right]^{\frac{1}{2}}\mathrm{d}\xi \tag{A.16}$$

with $G(\xi, \theta)$ given by equation (15). Numerical values for $H(n, \theta)$ are listed in Table 3, with $H(n, 1) = 2{\cdot}06$ corresponding to an isothermal gas. It is seen that the band absorptance as given by equation (A.15) is an extremely weak function of the temperature profile, such that equation (A.14) should indeed be a valid approximation to equation (A.12).

Table 3. *Numerical values of* $H(n, \theta)$

	θ		
n	1	3/2	2
−1	2·06	2·07	2·08
1	2·06	2·06	2·09
4	2·06	2·06	2·07

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* Following the same reasoning as given by Chan and Tien [3], such a hypothetical band will, in the square-root limit, result in a greater dependency of A upon temperature than for a fundamental band with $\beta \sim T^{-\frac{1}{2}}$

UNE FORMULATION D'ABSORPTANCE DE BANDE POUR UN RAYONNEMENT GAZEUX ISOTHERME

Résumé— On présente une méthode pour laquelle les corrélations pour l'absorptance totale de bande des bandes de vibration–rotation, applicables aux gaz isothermes, peut être étendue pour prédire l'absorptance totale de bande pour des gaz non isothermes. La méthode est analogue à l'approximation à deux paramètres de Curtis–Godson, et consiste à formuler l'absorptance totale de bande en fonction de trois paramètres qui décrivent un gaz isotherme équivalent. Ces trois paramètres sont choisis de telle façon que l'absorptance nonisotherme de bande est correcte dans les limites linéaires, en racine carrée et logarithmique. On montre que la formulation est en accord raisonnable avec les résultats expérimentaux et d'autres prévisions analytiques.

EINE BANDABSORPTIONSFORMULIERUNG FÜR ISOTHERME GASSTRAHLUNG

Zusammenfassung—Es wird eine Methode angegeben für die Erweiterung von Korrelationen für die Gesamtbandabsorption von Vibrations-Rotationsbanden von isothermen Gasen auf die Gesamtbandabsorption nicht-isothermer Gase. Die Methode ist analog, der zwei-parametrigen Curtis-Godson Näherung lund besteht darin, die Gesamtbandabsorption in Ausdrücken von drei Parametern, die ein äquivalentes Gas beschreiben, zu formulieren. Diese drei Parameter sind so gewählt, dass die nicht-isotherme Bandabsorption korrekt ist für lineare und logarithmische Grenzen, sowie für die der Quadratwurzel. Die Formulierung erweist sich in befriedigender Übereinstimmung mit Versuchsergebnissen und anderen analytischen Behandlungen.

О СПЕКТРЕ ПОГЛОЩЕНИЯ ДЛЯ НЕИЗОТЕРМИЧЕСКОГО ГАЗОВОГО ИЗЛУЧЕНИЯ

Аннотация—Приводится метод, согласно которому корреляции полного спектра поглощения колебательно-вращательного спектра, применимые для случая изотермических газов, могут использоваться для расчета полного спектра поглощения неизотермических газов. Метод аналогичен двухпараметрической аппроксимации Кертиса-Годсона и состоит в представлении общего спектра поглощения с помощью трех параметров, описывающих эквивалентный изотермический газ. Эти три параметра выбираются так, чтобы неизотермический спектр поглощения мог обеспечить точность в линейных, логарифмических и квадратного корня приближениях. Показано, что формулировка удовлетворительно согласуется с экспериментальными данными и другими теоретическими расчетами.